Michael Naper
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AOE 4134
Homework 8

Problem 1 A satellite is on an orbit with periapsis radius of 2 DU and eccentricity of 2. What is the satellite's velocity when it passes through $\nu = \pi/2$?

Solution:

$$r_p = \frac{p}{1+e}$$
$$p = r_p(1+e)$$

$$\vec{v} = \sqrt{\frac{\mu}{p}}\left[(-\sin\nu)\hat{P} + (e+\cos\nu)\hat{Q}\right]$$
$$= \sqrt{\frac{\mu}{r_p(1+e)}}\left[(-\sin\nu)\hat{P} + (e+\cos\nu)\hat{Q}\right]$$
$$= \sqrt{\frac{1}{2(1+2)}}\left[\left(-\sin\frac{\pi}{2}\right)\hat{P} + \left(2+\cos\frac{\pi}{2}\right)\hat{Q}\right]$$
$$= \frac{\sqrt{6}}{6}\left[-\hat{P} + 2\hat{Q}\right]\ {}^{DU}/_{TU}$$

$$|\vec{v}| = \frac{\sqrt{30}}{6}\ DU/TU$$

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Problem 2 Consider the orbit that connects the position vectors $\vec{r}_1 = 2\,\hat{J}\,DU$ and $\vec{r}_2 = 4\,\hat{K}\,DU$ with an orbit parameter of $p = 2.1149\,DU$. What is the short-way around time-of-flight for the orbit?

Solution:

$$k = r_1 r_2 (1 - \cos\Delta\nu) = 2 \cdot 4 \cdot (1 - \cos 90°) = 8$$

$$l = r_1 + r_2 = 2 + 4 = 6$$

$$m = r_1 r_2 (1 + \cos\Delta\nu) = 2 \cdot 4 \cdot (1 + \cos 90°) = 8$$

$$a = \frac{mkp}{(2m - l^2)p^2 + 2klp - k^2} = \frac{8 \cdot 8 \cdot 2.1149}{(2 \cdot 8 - 6^2)2.1149^2 + 2 \cdot 8 \cdot 6 \cdot 2.1149 - 8^2} = 2.73\,DU$$

$$f = 1 - \frac{r_2}{p}(1 - \cos\Delta\nu) = 1 - \frac{4}{2.1149}(1 - \cos 90°) = -0.891$$

$$\dot{f} = \sqrt{\frac{\mu}{p}} \tan\left(\frac{\Delta\nu}{2}\right)\left[\frac{1 - \cos\Delta\nu}{p} - \frac{1}{r_1} - \frac{1}{r_2}\right] = \sqrt{\frac{1}{2.1149}} \tan\left(\frac{90°}{2}\right)\left[\frac{1 - \cos 90°}{2.1149} - \frac{1}{2} - \frac{1}{4}\right] = -0.1906$$

$$g = \frac{r_1 r_2 \sin\Delta\nu}{\sqrt{\mu p}} = \frac{2 \cdot 4 \cdot \sin 90°}{\sqrt{1 \cdot 2.1149}} = 5.50$$

$$\sin\Delta E = -\frac{r_1 r_2 \dot{f}}{\sqrt{\mu a}} = -\frac{2 \cdot 4 \cdot (-0.1906)}{\sqrt{1 \cdot 2.73}} = 0.923$$

$$\cos\Delta E = 1 - \frac{r_1(1 - f)}{a} = 1 - \frac{2 \cdot \big(1 - (-0.891)\big)}{2.73} = -0.385$$

$$\Delta E = \tan^{-1}\left(\frac{\sin\Delta E}{\cos\Delta E}\right) = \tan^{-1}\left(\frac{0.923}{-0.385}\right) = 112.7° = 1.966\,rad$$

$$\Delta t = g + \sqrt{\frac{a^3}{\mu}}(\Delta E - \sin\Delta E) = 5.50 + \sqrt{\frac{2.73^3}{1}}(1.966 - \sin 1.966) = 10.21\,TU$$

$\Delta t = 10.21\,TU$

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Problem 3 A spacecraft is initially on a circular orbit of radius 1.2 DU and performs a co-planar transfer to a circular orbit of radius 2 DU. The transfer starts at periapsis and intersects the final circular orbit at a true anomaly of 90°. What is the total Δv required to perform the transfer?

Solution:

$$v_{cs1} = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{1}{1.2}} = 0.913\ DU/TU$$

$$r_1 = \frac{a_t(1-e_t^2)}{(1+e_t\cos\nu_1)} \quad \rightarrow \quad 1.2 = \frac{a_t(1-e_t^2)}{(1+e_t\cos 0°)} = a_t(1-e_t)$$

$$r_2 = \frac{a_t(1-e_t^2)}{(1+e_t\cos\nu_2)} \quad \rightarrow \quad 2 = \frac{a_t(1-e_t^2)}{(1+e_t\cos 90°)} = a_t(1-e_t^2)$$

$$a_t = 3.6\ DU$$

$$e_t = 0.667$$

$$p_t = a_t(1-e_t^2) = 3.6(1-0.667^2) = 2\ DU$$

$$\mathcal{E}_t = -\frac{\mu(1-e_t^2)}{2p_t} = -\frac{1\cdot(1-0.667^2)}{2\cdot 2} = -0.1389\ DU^2/TU^2$$

$$\Delta v_1 = \sqrt{2\left(\frac{\mu}{r_1}+\mathcal{E}_t\right)} - v_{cs1} = \sqrt{2\left(\frac{1}{1.2}-0.1389\right)} - 0.913 = 0.266\ DU/TU$$

$$v_2 = \sqrt{\mu\left(\frac{2}{r_2}-\frac{1}{a_t}\right)} = \sqrt{1\left(\frac{2}{2}-\frac{1}{3.6}\right)} = 0.850\ DU/TU$$

$$\phi_2 = \tan^{-1}\left(\frac{e_t\sin\nu_2}{1+e_t\cos\nu_2}\right) = \tan^{-1}(e_t) = 33.7$$

$$v_{cs2} = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{1}{2}} = 0.707\ DU/TU$$

$$\Delta v_2 = \sqrt{v_2^2 + v_{cs2}^2 - 2v_2 v_{cs2}\cos\phi_2} = \sqrt{0.850^2 + 0.707^2 - 2\cdot 0.850\cdot 0.707\cdot\cos 33.7°} = 0.471\ DU/TU$$

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2 = 0.266 + 0.471 = 0.737\ DU/TU$$

$$\boxed{\Delta v = 0.737\ DU/TU}$$

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Problem 4 A satellite is on an orbit with $a = 2\ DU$ and $e = 0.25$. If the spacecraft is initially at a true anomaly of $90°$, what is its true anomaly after $10\ TU$?

<u>Solution:</u>

$$E_0 = \cos^{-1}\left(\frac{e + \cos\nu_0}{1 + e\cos\nu_0}\right) = \cos^{-1}\left(\frac{0.25 + \cos 90°}{1 + 0.25\cos 90°}\right) = \cos^{-1}(0.25) = 1.318\ rad = 75.5°$$

$$M = n(t - t_0) = (E - e\sin E) - (E_0 - e\sin E_0) \quad \rightarrow \quad E - 0.25\sin E = \sqrt{\frac{1}{2^3}}(10) + [1.318 - 0.25\sin(1.318)] = 4.61\ rad$$

$$E = 4.38\ rad = 251°$$

$$\nu = \cos^{-1}\left(\frac{e - \cos E}{e\cos E - 1}\right) = \cos^{-1}\left(\frac{0.25 - \cos 4.38}{0.25\cos 4.38 - 1}\right) = 4.14\ rad = 237°$$

$\nu = 237°$

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Problem 5 An Earth satellite is to be launched directly into an orbit with $\Omega = 90°$ and $i = 50°$. The latitude of the launch site is $30°$. If the local sidereal time of the launch site is $180°$, how much time must pass before the next launch opportunity?

Solution:

$$\gamma = \sin^{-1}\left(\frac{\cos i}{\cos L_s}\right) = \sin^{-1}\left(\frac{\cos 50°}{\cos 30°}\right) = 47.9°$$

$$\delta = \cos^{-1}\left(\frac{\cos \gamma}{\sin i}\right) = \cos^{-1}\left(\frac{\cos 47.9°}{\sin 50°}\right) = 29.0°$$

$$LWST_{AN} = \Omega + \delta = 90° + 29.0° = 119.0°$$

$$LWST_{DN} = \Omega + (180° - \delta) = 90° + (180° - 29.0°) = 241°$$

$$\Delta\theta = LWST_{DN} - \theta = 241° - 180° = 61.0° = 1.065\ rad$$

$$\Delta t = \frac{\Delta\theta}{\omega_{Earth}} = \frac{1.065\ rad}{7.292 \times 10^{-5}\ \frac{rad}{s}} = 14\,610\ s = 4.06\ hr$$

$\Delta t = 4.06\ hr$

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Problem 6 A spacecraft in a circular orbit of radius $2\ DU$ is to rendezvous with another spacecraft in a circular orbit or radius $4\ DU$. A Hohmann transfer is used to perform the rendezvous. What is the lead angle, α_{lead}, required to perform the rendezvous maneuver?

Solution:

$$a_t = \frac{r_1 + r_2}{2} = \frac{2+4}{2} = 3\ DU$$

$$TOF = \pi\sqrt{\frac{a_t^3}{\mu}} = \pi\sqrt{3^3} = 16.32\ TU$$

$$v_{target} = \sqrt{\frac{\mu}{r_2}} = \sqrt{\frac{1}{4}} = 0.5\ DU/TU$$

$$\omega_{target} = \frac{v_{target}}{r_2} = \frac{0.5}{4} = 0.125\ rad/TU$$

$$\alpha_{lead} = \omega_{target} TOF = 0.125 \cdot 16.32 = 2.04\ rad = 116.9°$$

$$\boxed{\alpha_{lead} = 116.9°}$$

Problem 7 Two spacecraft, $\mathcal{A}$ and $\mathcal{B}$, are on the same circular orbit of radius $4\ DU$. Spacecraft $\mathcal{A}$ is "ahead" of spacecraft $\mathcal{B}$ by an angle of $30°$. What is the total Δv required for $\mathcal{B}$ to rendezvous with $\mathcal{A}$ using an interior phasing orbit?

Solution:

$$v_{cs1} = v_{cs2} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{1}{4}} = 0.5\ DU/TU$$

$$\omega_{target} = \frac{v_{target}}{r} = \frac{0.5}{4} = 0.125\ rad/TU$$

$$\phi_{travel} = 2\pi - \phi_{initial} = 2\pi - 0.524 = 5.76\ rad$$

$$a_{phasing} = \sqrt[3]{\mu\left(\frac{\phi_{travel}}{2\pi\omega_{target}}\right)^2} = \sqrt[3]{\left(\frac{5.76}{2\pi \cdot 0.125}\right)^2} = 3.77\ DU$$

$$\mathcal{E}_{phasing} = -\frac{\mu}{2a_{phasing}} = -\frac{1}{2 \cdot 3.77} = -0.1325\ DU^2/TU^2$$

$$|\Delta v_1| = \left|\sqrt{2\left(\frac{\mu}{r} + \mathcal{E}_t\right)} - v_{cs1}\right| = \left|\sqrt{2\left(\frac{1}{4} - 0.1325\right)} - 0.5\right| = 0.01516\ DU/TU$$

$$|\Delta v_2| = |\Delta v_1| = 0.01516\ DU/TU$$

$$|\Delta v_{tot}| = |\Delta v_1| + |\Delta v_2| = 2(0.01516) = 0.0303\ DU/TU$$

$$\Delta v = 0.0303\ DU/TU$$

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Problem 8 A spacecraft is on an Earth-to-Mars transfer. The heliocentric portion of the transfer begins at periapsis and intersects the orbit of Mars at a true anomaly of 150°. What is the velocity of the spacecraft relative to Mars when it intersects the orbit of Mars?

Solution:

$$\mu_{Sun} = 1.326 \times 10^{11}\ \frac{km^3}{s^2}$$

$$r_{Earth/Sun} = 1\ 496 \times 10^5\ km$$

$$r_{Mars/Sun} = 2\ 279.4 \times 10^5\ km$$

$$r_1 = \frac{a_t(1-e_t^2)}{(1+e_t\cos\nu_1)} \quad \rightarrow \quad 1\ 496 \times 10^5 = \frac{a_t(1-e_t^2)}{(1+e_t\cos 0°)} = a_t(1-e_t)$$

$$r_2 = \frac{a_t(1-e_t^2)}{(1+e_t\cos\nu_2)} \quad \rightarrow \quad 2\ 279.4 \times 10^5 = \frac{a_t(1-e_t^2)}{(1+e_t\cos 150°)}$$

$$a_t = 1\ 932 \times 10^5\ km$$

$$e_t = 0.226$$

$$p_t = a_t(1-e_t^2) = (1\ 932 \times 10^5)(1-0.226^2) = 1\ 834 \times 10^5\ km$$

$$\vec{v}_{t2} = \sqrt{\frac{\mu_{Sun}}{p_t}}\left[(-\sin\nu_{Mars})\hat{P} + (e_t + \cos\nu_{Mars})\hat{Q}\right] = \sqrt{\frac{(1.326 \times 10^{11})}{(1\ 834 \times 10^5)}}\left[(-\sin 150°)\hat{P} + (0.226 + \cos 150°)\hat{Q}\right]$$

$$= -13.44\hat{P} - 17.22\hat{Q}\ \frac{km}{s}$$

$$\vec{v}_{Mars} = \sqrt{\frac{\mu_{Sun}}{p_{Mars}}}\left[(-\sin\nu_{Mars})\hat{P} + (e_{Mars} + \cos\nu_{Mars})\hat{Q}\right] = \sqrt{\frac{(1.326 \times 10^{11})}{(2\ 279.4 \times 10^5)}}\left[(-\sin 150°)\hat{P} + (0 + \cos 150°)\hat{Q}\right]$$

$$= -12.06\hat{P} - 20.9\hat{Q}\ \frac{km}{s}$$

$$\vec{v}_{spacecraft/Mars} = \vec{v}_{t2} - \vec{v}_{Mars} = \left(-13.44\hat{P} - 17.22\hat{Q}\right) - \left(-12.06\hat{P} - 20.9\hat{Q}\right) = -1.385\hat{P} + 3.67\hat{Q}\ \frac{km}{s}$$

$$\left|\vec{v}_{spacecraft/Mars}\right| = 3.96\ \frac{km}{s}$$

$v = 3.96\ km/s$

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Problem 9 A spacecraft is initially in a circular orbit about Earth with an altitude of $400\ km$. The spacecraft is to perform a Δv such that when it leaves Earth's sphere of influence it has a heliocentric velocity of $32\ km/s$ and a heliocentric fight-path angle of $0°$. What is the required Δv?

Solution:

$$v_p = \sqrt{\frac{\mu_{Earth}}{r_p}} = \sqrt{\frac{3.986 \times 10^5}{6\,378 + 400}} = 7.67\ \frac{km}{s}$$

$$v_{Earth/Sun} = \sqrt{\frac{\mu_{Sun}}{r_{Earth/Sun}}} = \sqrt{\frac{1.326 \times 10^{11}}{1\,496 \times 10^5}} = 29.8\ \frac{km}{s}$$

$$v_{\infty,Earth} = \sqrt{v_{Earth/Sun}^2 + v_{sc/Sun}^2 - 2v_{Earth/Sun}v_{sc/Sun}\cos\phi} = \sqrt{29.8^2 + 32^2 - 2 \cdot 29.8 \cdot 32 \cdot \cos 0°} = 2.23\frac{km}{s}$$

$$\mathcal{E}_t = \frac{v_{\infty,Earth}^2}{2} = \frac{2.23^2}{2} = 2.48\ \frac{km^2}{s^2}$$

$$v_0 = \sqrt{2\left(\frac{\mu_{Earth}}{r_p} + \mathcal{E}_t\right)} = \sqrt{2\left(\frac{3.986 \times 10^5}{6\,378 + 400} + 2.48\right)} = 11.07\ \frac{km}{s}$$

$$\Delta v = \left|v_0 - v_p\right| = 11.07 - 7.67 = 3.40\frac{km}{s}$$

$\Delta v = 3.40\ km/s$

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Problem 10 A spacecraft arrives at Mars' sphere of influence with a heliocentric velocity of $22\ km/s$ and a heliocentric fight-path angle of $10°$. When the spacecraft reaches the periapsis of its Mars-centric arrival trajectory, at an altitude of $200\ km$, it performs a Δv to circularize its trajectory. What is the required Δv?

<u>Solution:</u>

$$v_p = \sqrt{\frac{\mu_{Mars}}{r_p}} = \sqrt{\frac{42\,828}{3\,397 + 200}} = 3.45\ \frac{km}{s}$$

$$v_{Mars/Sun} = \sqrt{\frac{\mu_{Sun}}{r_{Mars/Sun}}} = \sqrt{\frac{1.326 \times 10^{11}}{2\,279.4 \times 10^5}} = 24.1\ \frac{km}{s}$$

$$v_{\infty,Mars} = \sqrt{v_{Mars/Sun}^2 + v_{sc/Sun}^2 - 2 v_{Mars/Sun} v_{sc/Sun} \cos\phi} = \sqrt{24.1^2 + 22^2 - 2 \cdot 24.1 \cdot 22 \cdot \cos 10°} = 4.54 \frac{km}{s}$$

$$\mathcal{E}_t = \frac{v_{\infty,Mars}^2}{2} = \frac{4.54^2}{2} = 10.31\ \frac{km^2}{s^2}$$

$$v_0 = \sqrt{2\left(\frac{\mu_{Mars}}{r_p} + \mathcal{E}_t\right)} = \sqrt{2\left(\frac{42\,828}{3\,397 + 400} + 10.31\right)} = 6.57\ \frac{km}{s}$$

$$\Delta v = \left|v_0 - v_p\right| = 6.57 - 3.45 = 3.21 \frac{km}{s}$$

$\Delta v = 3.21\ km/s$