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Problem 1: Consider a biconvex airfoil in a free stream of Mach 2 where the upper and lower surfaces are given as:

$$\frac{y_u}{c} = 0.1\frac{x}{c}\left(1 - \frac{x}{c}\right)$$

$$\frac{y_l}{c} = -0.05\frac{x}{c}\left(1 - \frac{x}{c}\right)$$

(a) Calculate the pressure coefficient on both the upper and lower surface of the airfoil if it is flying at an angle of attack of 0°.

(b) Calculate the lift and drag coefficients on both the upper and lower surface of the airfoil if it is flying at an angle of attack of (i) 1° and (ii) 3°.

(c) What is the lift curve slope C_{L_α}?

Solution:

$$\alpha_c = \frac{1}{2}\frac{d(y_u + y_l)}{dx} = \frac{1}{2}\frac{d\left(^{y_u}/_c + ^{y_l}/_c\right)}{d(^x/_c)} = \frac{1}{2}\frac{d}{d(^x/_c)}\left[0.05\frac{x}{c}\left(1 - \frac{x}{c}\right)\right] = 0.025\left(1 - 2\frac{x}{c}\right)$$

$$\alpha_t = \frac{1}{2}\frac{d(y_u - y_l)}{dx} = \frac{1}{2}\frac{d\left(^{y_u}/_c - ^{y_l}/_c\right)}{d(^x/_c)} = \frac{1}{2}\frac{d}{d(^x/_c)}\left[0.15\frac{x}{c}\left(1 - \frac{x}{c}\right)\right] = 0.075\left(1 - 2\frac{x}{c}\right)$$

$$\overline{\alpha_c^2} = \frac{1}{c}\int_0^c \alpha_c^2\, dx = \frac{1}{c}\int_0^c \left[0.025\left(1 - 2\frac{x}{c}\right)\right]^2 dx = 0.208 \times 10^{-3}$$

$$\overline{\alpha_t^2} = \frac{1}{c}\int_0^c \alpha_t^2\, dx = \frac{1}{c}\int_0^c \left[0.075\left(1 - 2\frac{x}{c}\right)\right]^2 dx = 1.875 \times 10^{-3}$$

(a)

$$C_{p_u} = \frac{2\alpha_u}{\sqrt{M_\infty^2 - 1}} = \frac{2(-\alpha_0 + \alpha_c + \alpha_t)}{\sqrt{M_\infty^2 - 1}} = \frac{2\left[-\alpha_0 + 0.1\left(1 - 2\frac{x}{c}\right)\right]}{\sqrt{M_\infty^2 - 1}} = \frac{0.2\left(1 - 2\frac{x}{c}\right)}{\sqrt{(2)^2 - 1}} = \frac{\sqrt{3}}{15}\left(1 - 2\frac{x}{c}\right)$$

$$C_{p_l} = \frac{2\alpha_l}{\sqrt{M_\infty^2 - 1}} = \frac{2(\alpha_0 - \alpha_c + \alpha_t)}{\sqrt{M_\infty^2 - 1}} = \frac{2\left[\alpha_0 + 0.05\left(1 - 2\frac{x}{c}\right)\right]}{\sqrt{M_\infty^2 - 1}} = \frac{0.1\left(1 - 2\frac{x}{c}\right)}{\sqrt{(2)^2 - 1}} = \frac{\sqrt{3}}{30}\left(1 - 2\frac{x}{c}\right)$$

(b)

$$C_L = \frac{4\alpha_0}{\sqrt{M_\infty^2 - 1}} = \frac{4\alpha_0}{\sqrt{(2)^2 - 1}} = \frac{4\sqrt{3}}{3}\alpha_0$$

$$C_D = \frac{4\left(\alpha_0^2 + \overline{\alpha_c^2} + \overline{\alpha_t^2}\right)}{\sqrt{M_\infty^2 - 1}} = \frac{4(\alpha_0^2 + 2.08 \times 10^{-3})}{\sqrt{(2)^2 - 1}} = \frac{4\sqrt{3}}{3}(\alpha_0^2 + 2.08 \times 10^{-3})$$

(i) $\alpha_0 = 1°$:

$$C_L = \frac{4\sqrt{3}}{3}\left(1° \cdot \frac{\pi}{180°}\right) = 0.0403$$

$$C_D = \frac{4\sqrt{3}}{3}\left[\left(1° \cdot \frac{\pi}{180°}\right)^2 + 2.08 \times 10^{-3}\right] = 0.00551$$

(ii) $\alpha_0 = 3°$:

$$C_L = \frac{4\sqrt{3}}{3}\left(3° \cdot \frac{\pi}{180°}\right) = 0.1209$$

$$C_D = \frac{4\sqrt{3}}{3}\left[\left(3° \cdot \frac{\pi}{180°}\right)^2 + 2.08 \times 10^{-3}\right] = 0.01114$$

(c)

$$C_{L_\alpha} = \frac{4\sqrt{3}}{3}$$

Problem 2: You are designing a supersonic airfoil to fly at a Mach number of 2 and an angle of attack of 2°. The airfoil is a double wedge with the max thickness in the center. Using supersonic thin airfoil theory,

(a) Calculate the lift coefficient of the airfoil at the design conditions.
(b) Determine the thickness to chord ratio, t/c, required such that the airfoil will have a drag coefficient no larger than $C_D = 0.1$ at the design conditions.
(c) Calculate the pressure coefficient of the airfoil on the upper and lower surfaces at the design conditions if the thickness is $t/c = 0.1$.
(d) Calculate the lift and drag coefficients using shock-expansion theory where the airfoil thickness is $t/c = 0.1$.
(e) Comment on the difference in results of the lift coefficient calculated with shock-expansion theory and with supersonic thin airfoil theory. In your response, discuss if and why changing the thickness of the double wedge will change the lift coefficient in (i) supersonic thin airfoil theory and (ii) shock expansion theory.

Solution:

$$\alpha_c = 0$$

$$\alpha_t = \begin{cases} \frac{t}{c}, & 0 < \frac{x}{c} < 0.5 \\ -\frac{t}{c}, & 0.5 < \frac{x}{c} < 1 \end{cases}$$

$$\overline{\alpha_c^2} = 0$$

$$\overline{\alpha_t^2} = \frac{1}{c}\int_0^c \alpha_t^2 \, dx = \frac{1}{c}\int_0^{c/2} \left(\frac{t}{c}\right)^2 dx + \frac{1}{c}\int_{c/2}^{c} \left(-\frac{t}{c}\right)^2 dx = \left(\frac{t}{c}\right)^2$$

(a)

$$C_L = \frac{4\alpha_0}{\sqrt{M_\infty^2 - 1}} = \frac{4\left(2° \cdot \frac{\pi}{180°}\right)}{\sqrt{(2)^2 - 1}} = 0.0806$$

(b)

$$C_D = \frac{4\left(\alpha_0^2 + \overline{\alpha_c^2} + \overline{\alpha_t^2}\right)}{\sqrt{M_\infty^2 - 1}} = \frac{4\left[\left(2° \cdot \frac{\pi}{180°}\right)^2 + \left(\frac{t}{c}\right)^2\right]}{\sqrt{(2)^2 - 1}} = \frac{4\sqrt{3}}{3}\left[\left(\frac{\pi}{90}\right)^2 + \left(\frac{t}{c}\right)^2\right]$$

$$\frac{t}{c} = \sqrt{\frac{\sqrt{3}}{4} C_D - \left(\frac{\pi}{90}\right)^2} = \sqrt{\frac{\sqrt{3}}{4}(0.1) - \left(\frac{\pi}{90}\right)^2} = 0.205$$

(c)

$$C_{p_u} = \frac{2\alpha_u}{\sqrt{M_\infty^2 - 1}} = \frac{2(-\alpha_0 + \alpha_c + \alpha_t)}{\sqrt{M_\infty^2 - 1}} = \begin{cases} \dfrac{2\left[-\left(2° \cdot \frac{\pi}{180°}\right) + (0.1)\right]}{\sqrt{(2)^2 - 1}} = 0.0752, & 0 < \dfrac{x}{c} < 0.5 \\ \dfrac{2\left[-\left(2° \cdot \frac{\pi}{180°}\right) - (0.1)\right]}{\sqrt{(2)^2 - 1}} = -0.1558, & 0.5 < \dfrac{x}{c} < 1 \end{cases}$$

$$C_{p_l} = \frac{2\alpha_l}{\sqrt{M_\infty^2 - 1}} = \frac{2(\alpha_0 - \alpha_c + \alpha_t)}{\sqrt{M_\infty^2 - 1}} = \begin{cases} \dfrac{2\left[\left(2° \cdot \frac{\pi}{180°}\right) + (0.1)\right]}{\sqrt{(2)^2 - 1}} = 0.1558, & 0 < \dfrac{x}{c} < 0.5 \\ \dfrac{2\left[\left(2° \cdot \frac{\pi}{180°}\right) - (0.1)\right]}{\sqrt{(2)^2 - 1}} = -0.0752, & 0.5 < \dfrac{x}{c} < 1 \end{cases}$$

(d)

$$M_1 = 2, \quad \delta_{TL} = \tan^{-1}\left({}^{t}/_{c}\right) - 2° = \tan^{-1}(0.1) - 2° = 3.71° \qquad \rightarrow \qquad \beta_{TL} = 33.1°$$

$$M_{1_n} = M_1 \sin(\beta_{TL}) = (2)\sin(33.1°) = 1.093 \qquad \rightarrow \qquad M_{TL_n} = 0.9172, \qquad \frac{p_{TL}}{p_1} = 1.227$$

$$M_{TL} = \frac{M_{TL_n}}{\sin(\beta - \delta)} = \frac{(0.9172)}{\sin(33.1° - 3.71°)} = 1.867 \qquad \rightarrow \qquad \frac{p_{TL}}{p_{0_{TL}}} = 0.1570$$

$$p_{TL} = \frac{p_{TL}}{p_1} p_1 = 1.227 p_1$$

$$M_1 = 2, \quad \delta_{BL} = \tan^{-1}\left({}^{t}/_{c}\right) + 2° = \tan^{-1}(0.1) + 2° = 7.71° \qquad \rightarrow \qquad \beta_{BL} = 36.9°$$

$$M_{1_n} = M_1 \sin(\beta_{BL}) = (2)\sin(36.9°) = 1.201 \qquad \rightarrow \qquad M_{BL_n} = 0.8413, \qquad \frac{p_{BL}}{p_1} = 1.517$$

$$M_{BL} = \frac{M_{BL_n}}{\sin(\beta - \delta)} = \frac{(0.8413)}{\sin(36.9° - 7.71°)} = 1.724 \qquad \rightarrow \qquad \frac{p_{BL}}{p_{0_{BL}}} = 0.1953$$

$$p_{BL} = \frac{p_{BL}}{p_1} p_1 = 1.517 p_1$$

$$M_{TL} = 1.867, \quad \theta_{TR} = 2\tan^{-1}\left({}^{t}/_{c}\right) = 2\tan^{-1}(0.1) = 11.42°$$

$$\nu(M_{TR}) = \theta + \nu(M_{TL}) = 11.42° + \nu(1.867) = 34.156° \quad \rightarrow \quad M_{TR} = 2.30, \quad \frac{p_{TR}}{p_{0_{TR}}} = 0.07997$$

$$p_{TR} = \frac{p_{TR}}{p_{0_{TR}}}\frac{p_{0_{TL}}}{p_{TL}}\frac{p_{TL}}{p_1}p_1 = \frac{(0.07997)(1.227)}{(0.1570)}p_1 = 0.6250p_1$$

$$M_{BL} = 1.724, \quad \theta_{BR} = 2\tan^{-1}\left({}^{t}/_{c}\right) = 2\tan^{-1}(0.1) = 11.42°$$

$$\nu(M_{BR}) = \theta + \nu(M_{BL}) = 11.42° + \nu(1.724) = 29.818° \quad \rightarrow \quad M_{BR} = 2.13, \quad \frac{p_{BR}}{p_{0_{BR}}} = 0.1043$$

$$p_{BR} = \frac{p_{BR}}{p_{0_{BR}}}\frac{p_{0_{BL}}}{p_{BL}}\frac{p_{BL}}{p_1}p_1 = \frac{(0.1043)(1.517)}{(0.1953)}p_1 = 0.8102p_1$$

$$\begin{aligned} L &= p_{BL}\cos\delta_{BL}\frac{c}{2} + p_{BR}\cos\delta_{BR}\frac{c}{2} - p_{TL}\cos\delta_{TL}\frac{c}{2} - p_{TR}\cos\delta_{TR}\frac{c}{2} \\ &= \frac{1}{2}\{[(1.517) - (0.6250)]\ \cos(7.71°) + [(0.8102) - (1.227)]\cos(3.71°)\}cp_1 \\ &= 0.2340cp_\infty \end{aligned}$$

$$\begin{aligned} D &= p_{TL}\sin\delta_{TL}\frac{c}{2} + p_{BL}\sin\delta_{BL}\frac{c}{2} - p_{TR}\sin\delta_{TR}\frac{c}{2} - p_{BR}\sin\delta_{BR}\frac{c}{2} \\ &= \frac{1}{2}\{[(1.517) - (0.6250)]\sin(7.71°) + [(1.227) - (0.8102)]\sin(3.71°)\}cp_1 \\ &= 0.0733cp_\infty \end{aligned}$$

$$C_L = \frac{L/c}{\frac{1}{2}\gamma p_\infty M_\infty^2} = \frac{(0.2340cp_\infty)/c}{\frac{1}{2}(1.4)p_\infty(2)^2} = 0.0836$$

$$C_D = \frac{D/c}{\frac{1}{2}\gamma p_\infty M_\infty^2} = \frac{(0.0733cp_\infty)/c}{\frac{1}{2}(1.4)p_\infty(2)^2} = 0.0262$$

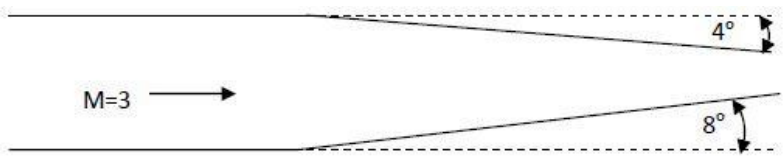
(e)

$$(C_L)_{thin\ airfoil\ theory} = 0.0806$$

$$(C_L)_{shock\ expansion\ theory} = 0.0836$$

The lift coefficient provided by shock expansion theory is only slightly higher than that provided by thin airfoil theory. Increasing the thickness-to-chord ratio for (i) supersonic airfoil theory *will not* change the lift coefficient, but for (ii) shock expansion theory *will* increase the lift coefficient as thickness will affect the angles the flow will encounter.

Problem 3: A diffuser has a constant area until the area shrinks by means of a sudden turn at an angle of $4°$ on the upper wall and $8°$ on the lower wall. The Mach number in the diffuser before the turn is $M = 3$ and the pressure is $200\ kPa$.



(a) Determine the shock angle, Mach number, and pressure on the upper wall just after the $4°$ turn.
(b) Determine the shock angle, Mach number, and pressure on the lower wall just after the $8°$ turn.
(c) Determine both reflected shock angles, the direction of the flow, and the Mach number and pressure in the regions after the reflected shocks.

Solution:

(a)

$$M_1 = 3, \quad \delta_{1_u} = 4° \qquad \rightarrow \qquad \beta_{1_u} = 22.35°$$

$$M_{1_{n_u}} = M_1 \sin(\beta_{1_u}) = (3)\sin(22.35°) = 1.141 \qquad \rightarrow \qquad M_{2_{n_u}} = 0.8813, \quad \left(\frac{p_2}{p_1}\right)_u = 1.352$$

$$M_{2_u} = \frac{M_{2_{n_u}}}{\sin(\beta_{1_u} - \delta_{1_u})} = \frac{(0.8813)}{\sin(22.35° - 4°)} = 2.799$$

$$p_{2_u} = \left(\frac{p_2}{p_1}\right)_u p_1 = (1.352)(200\ kPa) = 270\ kPa$$

(b)

$$M_1 = 3, \quad \delta_{1_l} = 8° \qquad \rightarrow \qquad \beta_{1_l} = 25.61°$$

$$M_{1_{n_l}} = M_1 \sin(\beta_{1_l}) = (3)\sin(25.61°) = 1.297 \qquad \rightarrow \qquad M_{2_{n_l}} = 0.7876, \quad \left(\frac{p_2}{p_1}\right)_l = 1.795$$

$$M_{2_l} = \frac{M_{2_{n_l}}}{\sin(\beta_{1_l} - \delta_{1_l})} = \frac{(0.7876)}{\sin(25.61° - 8°)} = 2.603$$

$$p_{2_l} = \left(\frac{p_2}{p_1}\right)_l p_1 = (1.795)(200\ kPa) = 359\ kPa$$

(c)

$\phi = \delta_{1_l} - \delta_{1_u} = 8° - 4° = 4°$

$M_{2_u} = 2.799, \quad \delta_{2_u} = \delta_{1_u} + \phi = 4° + 4° = 8° \quad \rightarrow \quad \beta_{2_u} = 27.16°$

$M_{2_{n_u}} = M_{2_u} \sin(\beta_{2_u}) = (2.799)\sin(27.16°) = 1.278 \quad \rightarrow \quad M_{3_{n_u}} = 0.7976, \quad \left(\frac{p_3}{p_2}\right)_u = 1.738$

$\left(\frac{p_3}{p_1}\right)_u = \left(\frac{p_3}{p_2}\right)_u \left(\frac{p_2}{p_1}\right)_u = (1.738)(1.352) = 2.350$

$M_{2_l} = 2.603, \quad \delta_{2_l} = \delta_{1_l} - \phi = 8° - 4° = 4° \quad \rightarrow \quad \beta_{2_l} = 25.58°$

$M_{2_{n_l}} = M_{2_l} \sin(\beta_{2_l}) = (2.603)\sin(25.58°) = 1.124 \quad \rightarrow \quad M_{3_{n_l}} = 0.8936, \quad \left(\frac{p_3}{p_2}\right)_l = 1.307$

$\left(\frac{p_3}{p_1}\right)_l = \left(\frac{p_3}{p_2}\right)_l \left(\frac{p_2}{p_1}\right)_l = (1.307)(1.795) = 2.346$

$\frac{p_{3_u}}{p_{3_l}} = \left(\frac{p_3}{p_1}\right)_u \left(\frac{p_1}{p_3}\right)_l = \frac{(2.350)}{(2.346)} = 1.002 \quad \rightarrow \quad \phi\ OK$

$M_{3_u} = \frac{M_{3_{n_u}}}{\sin(\beta_{2_u} - \delta_{2_u})} = \frac{(0.7976)}{\sin(27.16° - 8°)} = 2.430$

$p_{3_u} = \left(\frac{p_3}{p_1}\right)_u p_1 = (2.350)(200\ kPa) = 470\ kPa$

$M_{3_l} = M_{3_u} = 2.430$

$p_{3_l} = p_{3_u} = 470\ kPa$

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2011.04.14 **Homework 9**

Problem 4: Consider a supersonic flightier jet flying at a Mach number of 2.6 at an altitude where the pressure is $60\ kPa$. The nose of the aircraft is a cone shape with a cone angle of $28°$. The supersonic flow generates a conical shock wave on the nose of the aircraft.

(a) Determine the Mach number and pressure in the region after the shock.
(b) Determine the Mach number of the flow on the cone surface.
(c) Determine the pressure coefficient on the cone.

Solution:

(a)

$$M_1 = 2.6, \quad \delta = 14° \qquad \rightarrow \qquad \beta = 34.75°$$

$$M_{1_n} = M_1 \sin(\beta) = (2.6)\sin(34.75°) = 1.482 \qquad \rightarrow \qquad M_{2_n} = 0.7076, \quad \frac{p_2}{p_1} = 2.395$$

$$M_2 = \frac{M_{2_n}}{\sin(\beta - \delta)} = \frac{(0.7076)}{\sin(34.75° - 14°)} = 1.997$$

$$p_2 = \frac{p_2}{p_1} p_1 = (2.395)(60\ kPa) = 143.7\ kPa$$

(b)

$$M_1 = 2.6, \quad \sigma = 14° \qquad \rightarrow \qquad C_{p_c} = 0.18, \qquad 1 - \frac{1}{M_c} = 0.57$$

$$1 - \frac{1}{M_c} = 0.57 \quad \rightarrow \quad M_c = 2.33$$

(c)

$$C_{p_c} = \frac{p_c - p_1}{\frac{1}{2}\gamma p_1 M_1^2} = 0.18 \quad \rightarrow \quad p_c = \frac{1}{2}(0.18)\gamma p_1 M_1^2 + p_1 = \frac{1}{2}(0.18)(1.4)(60\ kPa)(2.6)^2 + (60\ kPa) = 111.1056\ kPa$$