NAME *Michael Naper*

Math 4564 Fall '09 Test 2

1. Given the partial differential equation
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 6\frac{\partial u}{\partial x} = 0,$$ set $u(x,t) = X(x)T(t)$. Use the separation of variables method (That is, step one only) to find the two differential equations satisfied by $X(x)$ and $T(t)$.

$$u(x,t) = X(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t) \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \qquad \frac{\partial u}{\partial x} = X'(x)T(t)$$

$$X(x)T''(t) - X''(x)T(t) + 6X'(x)T(t) = 0$$

$$X(x)T''(t) - [X''(x) - 6X'(x)]T(t) = 0$$

$$X(x)T''(t) = [X''(x) - 6X'(x)]T(t)$$

$$\frac{T''(t)}{T(t)} = \frac{X''(x) - 6X'(x)}{X(x)}$$

$$\frac{T''(t)}{T(t)} = -\lambda \Rightarrow T''(t) + \lambda T(t) = 0$$

$$\frac{X''(x) - 6X'(x)}{X(x)} = -\lambda \Rightarrow X''(x) - 6X'(x) + \lambda X(x) = 0$$

2. Given the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \; t > 0$$

with $u(0,t) = 30$, $\frac{\partial u}{\partial x}(L,t) = 50$.

What form does the solution to this problem have?

$$u(x,t) = v(x) + w(x,t)$$

$$w(x,t) = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{(2n-1)\pi}{2L}\right)x\right] \exp\left[-\left(\frac{(2n-1)\pi}{2L}\right)^2 t\right]$$

$$f(x) = w(x,0) = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{(2n-1)\pi}{2L}\right)x\right] = u(x,0) - v(x)$$

$$u(0,t) = 30 = w(0,t) + v(0)$$

$$30 = 0 + v(0)$$

$$v(0) = 30$$

$$\frac{\partial u}{\partial x}(L,t) = 50 = \frac{\partial w}{\partial x}(L,t) + v'(L)$$

$$50 = 0 + v'(L)$$

$$v'(L) = 50$$

$$\therefore u(x,t) = 50x + 30 + \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{(2n-1)\pi}{2L}\right)x\right] \exp\left[-\left(\frac{(2n-1)\pi}{2L}\right)^2 t\right]$$

3. Given $X''(x) + k^2 X(x) = 0$, $k > 0$ with

$X'(0) = 0$, $X(\pi/2) = 0$

Find those values of k for which nonzero solutions $X_n(x)$ exist. Find the $X_n(x)$, $n = 1, 2, \ldots$

$X(x) = c_1 \cos(kx) + c_2 \sin(kx)$

$X'(x) = -kc_1 \sin(kx) + kc_2 \cos(kx)$

$X'(0) = 0 = kc_2$ $\qquad$ $X(\pi/2) = 0 = c_1 \cos(\pi/2\, k)$, $c_1 \neq 0$

$c_2 = 0$ $\qquad$ $0 = \cos(\pi/2\, k)$

$(2n-1)\frac{\pi}{2} = \frac{\pi}{2} k$

$k = 2n - 1$, $n \geq 1$

$X_n(x) = C_n \cos[(2n-1)x]$, $n = 1, 2, 3, \ldots$

4. Given $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$, $u(0,t) = u(\pi, t) = 0$

$u(x,0) = 4\sin(5x)$

$\frac{\partial u}{\partial t}(x,0) = -3\sin(7x)$

Choose a suitable form for the solution $u(x,t)$. Then choose the Fourier coefficients correctly so that the initial conditions are satisfied.

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\tfrac{n\pi}{L} t\right) + B_n \sin\left(\tfrac{n\pi}{L} t\right)\right] \sin\left(\tfrac{n\pi}{L} x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(nt) + B_n \sin(nt)\right] \sin(nx)$$

$$f(x) = u(x,0) = 4\sin(5x) = \sum_{n=1}^{\infty} A_n \sin(nx) \Rightarrow A_5 = 4$$

$$A_n = 0,\ n \neq 5$$

$$u(x,t) = 4\cos(5t)\sin(5x) + \sum_{n=1}^{\infty} B_n \sin(nt)\sin(nx)$$

$$\frac{\partial u}{\partial t}(x,t) = -20\sin(5t)\sin(5x) + \sum_{n=1}^{\infty} nB_n \cos(nt)\sin(nx)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = -3\sin(7x) = \sum_{n=1}^{\infty} nB_n \sin(nx) \Rightarrow B_7 = -\frac{3}{7}$$

$$B_n = 0,\ n \neq 7$$

$$u(x,t) = 4\cos(5t)\sin(5x) - \frac{3}{7}\sin(7t)\sin(7x)$$

5. Compute The Fourier sine series for $f(x) = x^2 - Lx$, $0 < x < L$. Does the Fourier series equal the function for $0 \le x \le L$?

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2}{L}\int_0^L (x^2 - Lx) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx - \frac{2}{L}\int_0^L Lx \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx - 2\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2}{L}\left[-\left(\frac{L}{n\pi}\right) x^2 \cos\left(\frac{n\pi x}{L}\right) + 2\left(\frac{L}{n\pi}\right)^2 x \sin\left(\frac{n\pi x}{L}\right) + 2\left(\frac{L}{n\pi}\right)^3 \cos\left(\frac{n\pi x}{L}\right)\right]_0^L - 2\left[-\left(\frac{L}{n\pi}\right) x \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$
$$= \frac{2}{L}\left[-\left(\frac{L}{n\pi}\right) L^2 \cos(n\pi) + 2\left(\frac{L}{n\pi}\right)^3 \cos(n\pi) - 2\left(\frac{L}{n\pi}\right)^3\right] - 2\left[-\left(\frac{L}{n\pi}\right) L \cos(n\pi)\right]$$
$$= (-1)^{n+1}\frac{2L^2}{n\pi} + \left[(-1)^n - 1\right]\frac{4L^2}{(n\pi)^3} + (-1)^n \frac{2L^2}{n\pi}$$
$$= \left[(-1)^n - 1\right]\frac{4L^2}{(n\pi)^3}$$
$$= \begin{cases} -\dfrac{8L^2}{(n\pi)^3}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$-\sum_{k=1}^{\infty} \frac{8L^2}{[(2k-1)\pi]^3} \sin\left[\frac{(2k-1)\pi x}{L}\right]$$

6. The isoperimetric inequality in 2 dimensions (which can be proved using Fourier series) says that if a smooth curve $\mathcal{C}$ in the xy plane encloses an area A and has perimeter P, then

$$A \le \frac{P^2}{4\pi}.$$

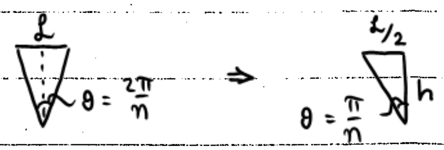
Furthermore, if $\frac{P^2}{4\pi} = A$, then $\mathcal{C}$ is a circle.

Compute $\frac{(P_N)^2}{4\pi A_N}$ for a N-sided regular polygon with P_N = its perimeter and A_N it's area.

Compute $\lim_{N\to\infty} \frac{(P_N)^2}{4\pi A_N}$ and interpret the answer geometrically.

Let G_n be an n-sided regular polygon with side-length ℓ.

$P_n(G_n) = n\ell$



$$A = \tfrac{1}{2}\left(\tfrac{\ell}{2}\right) h = \tfrac{1}{4}\ell h$$
$$h = \frac{\ell}{2\tan(\pi/n)} = \tfrac{1}{2}\ell\cot\left(\tfrac{\pi}{n}\right)$$
$$A = \tfrac{1}{8}\ell^2\cot\left(\tfrac{\pi}{n}\right)$$

$$A_n(G_n) = \tfrac{1}{4} n \ell^2 \cot\left(\tfrac{\pi}{n}\right)$$

$$\frac{P_n^2}{4\pi A_n} = \frac{(n\ell)^2}{n\pi \ell^2 \cot(\pi/n)} = \frac{1}{\pi} n \tan\left(\frac{\pi}{n}\right)$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{P_n^2}{4\pi A_n} &= \lim_{n\to\infty} \frac{1}{\pi} n \tan\left(\frac{\pi}{n}\right) \\
&= \lim_{n\to\infty} \frac{\sin(\pi/n)}{(\pi/n)\cos(\pi/n)} \\
&= \lim_{n\to\infty} \frac{\sin(\pi/n)}{(\pi/n)} \cdot \lim_{n\to\infty} \frac{1}{\cos(\pi/n)} \\
&= 1 \cdot \frac{1}{1} \\
&= 1
\end{aligned}$$

A circle can be thought of as a regular polygon with infinitely many sides, i.e., $n \to \infty$. Thus we have shown that the ratio $\frac{P_n^2}{4\pi A_n}$ is 1 for a circle, i.e., $\frac{P^2}{4\pi} = A$.

7. Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t) \text{ with}$$

the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = T_1, \quad \frac{\partial u}{\partial x}(L,t) = T_1$$

and the initial condition

$u(x,0) = T_2$, with T_1, T_2 constants

$T_1 > 0, T_2 > 0$.

Note that the steady state solution cannot be determined from the boundary conditions alone.

Find the exact solution.

$$u(x,t) = v(x) + w(x,t)$$

$$v(x) = Ax + B, \quad v'(0) = T_1, \quad v'(L) = T_1 \quad \Rightarrow \quad v(x) = T_1 x + B$$

$$w(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) \exp\left[-\left(\frac{n\pi}{L}\right)^2 t\right]$$

$$u(x,t) = T_1 x + B + \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) \exp\left[-\left(\frac{n\pi}{L}\right)^2 t\right]$$

$$\lim_{t\to\infty} u(x,t) = T_1 x + B + \frac{A_0}{2} \qquad f(x) = w(x,0) = T_2 - (T_1 x + B)$$

$$\frac{A_0}{2} = \frac{1}{L}\int_0^L \left[T_2 - (T_1 x + B)\right] dx$$
$$= \frac{1}{L}\left[T_2 x - \frac{1}{2}T_1 x^2 - Bx\right]_0^L$$

$$= T_2 - \frac{1}{2} T_1 L - B$$

$$\lim_{t\to\infty} u(x,t) = T_1 x + B + \left[T_2 - \frac{1}{2} T_1 L - B \right]$$

$$= \left(x - \frac{1}{2} L \right) T_1 + T_2$$

$$A_n = \frac{2}{L} \int_0^L \left(T_2 - B - T_1 x \right) \cos\left(\frac{n\pi}{L} x \right) dx$$

$$= \frac{2}{L} \left[\left(\frac{L}{n\pi} \right) \left(T_2 - B - T_1 x \right) \sin\left(\frac{n\pi}{L} x \right) \Big|_0^L - T_1 \left(\frac{L}{n\pi} \right)^2 \cos\left(\frac{n\pi}{L} x \right) \Big|_0^L \right]$$

$$= \frac{2}{L} \left[(-1)^{n+1} + 1 \right] T_1 \left(\frac{L}{n\pi} \right)^2$$

$$= \left[(-1)^{n+1} + 1 \right] \frac{2 T_1 L}{(n\pi)^2}$$

$$= \begin{cases} \frac{4 T_1 L}{(n\pi)^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$u(x,t) = \left[T_1 x + B \right] + \left[T_2 - \frac{1}{2} T_1 L - B \right]$$

$$+ \sum_{k=1}^{\infty} \frac{4 T_1 L}{[(2k-1)\pi]^2} \cos\left[\frac{(2k-1)\pi}{L} x \right] \exp\left(- \left[\frac{(2k-1)\pi}{L} \right]^2 t \right)$$

$$= \left(x - \frac{1}{2} L \right) T_1 + T_2 + \sum_{k=1}^{\infty} \frac{4 T_1 L}{[(2k-1)\pi]^2} \cos\left[\frac{(2k-1)\pi}{L} x \right] \exp\left(- \left[\frac{(2k-1)\pi}{L} \right]^2 t \right)$$

$$= \left(x + \frac{7}{2} L \right) T_1 + T_2 + \sum_{k=1}^{\infty} \frac{1}{[(2k-1)\pi]^2} \cos\left[\frac{(2k-1)\pi}{L} x \right] \exp\left(- \left[\frac{(2k-1)\pi}{L} \right]^2 t \right)$$

8. Let $u(x,t) = \sum_1^\infty A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$

be a solution to the vibrating string problem. Suppose that the string is further constained at $x = L/3$, so that $u(L/3, t) = 0$ for all t. What condition does this impose on which of the Fourier coefficients A_n? That is, for which n are conditions imposed on A_n and what are these conditions?

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$u\left(\frac{L}{3}, t\right) = 0 = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi}{L} \cdot \frac{L}{3}\right)$$
$$= \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi}{3}\right)$$

$$A_n = 0, \quad n \neq 3k, \quad k \in \mathbb{Z}$$